Uniqueness of Green functions in metric spaces, yes and no

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In Euclidean space $\mathbb{R}^n$, a result of Kichenassamy and Verón shows that the n-Laplace operator

$$\mathcal{L}_n u := \operatorname{div}(|\nabla u|^{n-2}\nabla u)$$

admits a unique global Green function, i.e., there is a unique, properly normalized singular solution which blows up to $+\infty$ at the origin and converges to $-\infty$ at infinity. The argument was later simplified and extended to the Carnot group by Balogh, Holopainen and Tyson, thus establishing uniqueness of global Green functions in the conformal case $p = n$ in this geometry.

The purpose of this talk is to discuss the uniqueness of Q-harmonic Green functions for $Q > 1$ in the setting of complete metric spaces (X, d, μ) equipped with an Ahlfors Q-regular Borel measure μ, and a Poincaré inequality. Over the past 20 years, many aspects of first-order calculus have been systematically developed in such spaces and spaces satisfying these conditions include Euclidean spaces, Riemannian manifolds, and many nonEuclidean spaces like sub-Riemannian manifolds such as the Heisenberg group, Gromov-Hausdorff limits of manifolds with lower Ricci curvature bounds, visual boundaries of certain hyperbolic buildings, etc. For our particular question about the uniqueness of Green functions, it turns out that the answer depends on the availability of an inner product on the generalized cotangent space of the metric space.

The talk is based on two joint works, one with Mario Bonk and Luca Capogna, and the other with Anders Björn, Jana Björn and Sylvester Eriksson-Bique.